J. Math. Anal. Appl. 378 (2011) 140–150

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Frames in generalized Fock spaces

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ARTICLE INFO

Article history:
Received 25 February 2010
Available online 28 December 2010
Submitted by Steven G. Krantz

Keywords:
Bargmann transform
Frames
Finite truncation of an operator
Fock space
Gabor frame
Hilbert space
Time-frequency analysis

ABSTRACT

Let A denote a real linear transformation on $\mathbb{C}^n$ which is symmetric and positive-definite relative to the real inner product $Re\langle z,w\rangle$, $z,w\in\mathbb{C}^n$. Let $\mathcal{F}_A(\mathbb{C}^n)$ denote the Fock space consisting of holomorphic functions on $\mathbb{C}^n$ which are square integrable with respect to the Gaussian measure $d\mu_A(z)=\frac{\sqrt{\det_{\mathbb{R}} A}}{\pi^n}e^{-Re\langle Az,z\rangle}$. For $w\in\mathbb{C}^n$, let $e_w^A(z)=e^A(z,w)=e^{-\frac{1}{2}Re\langle Aw,w\rangle}K_A(z,w)$, $z\in\mathbb{C}^n$, where K_A is the reproducing kernel for $\mathcal{F}_A(\mathbb{C}^n)$. The main aim of this paper is to show that there exist $a,b>0$ such that the set of functions $\{e^A_{ma+inb}\colon m,n\in\mathbb{Z}^n\}$ forms a frame in $\mathcal{F}_A$.

1. Introduction

Frames are generalizations of orthogonal bases. In signal processing, frames are used to encode and reconstruct sounds and other signals. Frames were studied by Duffin and Schaeffer in [6] in connection with non-uniform sampling of band limited functions. However, the theory of frames become popular in mathematical physics, sampling theory, time-frequency analysis and wavelet theory after the papers [3,5] were published. In particular, there are two types of extremely useful frames known as Gabor frames [13] (Weyl–Heisenberg frames) and affine frames (wavelet frames), which have been studied extensively for the last fifteen years. (See, for example, [2,8–12,14–16,18–20].) We refer to the paper of Heil and Walnut [17] and Gröchenig [16] for definitions and basic results of frames.

In 2006, Fabec, Ólafsson and Sengupta in [7] defined a generalized Fock space $\mathcal{F}_A(\mathbb{C}^n)$, where A is a real linear transform on $\mathbb{C}^n$, which is assumed to be symmetric, positive-definite relative to the real inner product $(\cdot,\cdot)$ on $\mathbb{C}^n$. (The real inner product $(\cdot,\cdot)$ is defined to be $(z,w)=Re\langle z,w\rangle$, $z,w\in\mathbb{C}^n$, where $\langle z,w\rangle$ denotes the usual Hermitian inner product on $\mathbb{C}^n$.) The space $\mathcal{F}_A(\mathbb{C}^n)$ is defined as the Hilbert space of holomorphic functions $F:\mathbb{C}^n\to\mathbb{C}$ such that

$$\|F\|_A^2:=\int_{\mathbb{C}^n}\left|F(z)\right|^2 d\mu_A(z)<\infty,$$

where $d\mu_A(z)=\pi^{-n}\sqrt{\det_{\mathbb{R}} A}\,e^{-Re\langle Az,z\rangle}\,dx\,dy$, $z=x+iy$, $x,y\in\mathbb{R}^n$.

The space $\mathcal{F}_A(\mathbb{C}^n)$ is a reproducing kernel Hilbert space with reproducing kernel $K_A(z,w)$ given by

$$K_A(z,w)=C_A^{-2}e^{\frac{1}{2}\overline{\langle Kz,z\rangle}}e^{\langle Hz,w\rangle}e^{\frac{1}{2}\langle Kw,w\rangle},\tag{1.1}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.12.049

where $C_A = (\frac{\det_{\mathbb{R}}(A)}{\det_{\mathbb{R}}(H)})^{\frac{1}{4}}$, $A = H + K$,

$$H := \frac{A + J^{-1}AJ}{2} \quad \text{and} \quad K := \frac{A - J^{-1}AJ}{2}, \tag{1.2}$$

with $J : \mathbb{C}^n \to \mathbb{C}^n$ defined by $Jz = iz$, considering as a real linear transformation. For details, see [7].

As A is symmetric and positive-definite relative to the real inner product space $(\cdot,\cdot)$, H becomes self-adjoint, positive-definite with respect to the inner product $\langle\cdot,\cdot\rangle$ and K is conjugate linear with respect to the inner product $\langle\cdot,\cdot\rangle$. In fact, H is invertible.

The generalized Segal–Bargmann transform B_A is defined to be

$$B_A f(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \frac{(\det H)^{\frac{3}{4}}}{(\det_{\mathbb{R}} A)^{\frac{1}{4}}} e^{\frac{1}{2}(\langle Hz,\bar{z}\rangle + \langle z, Kz\rangle)} \int\limits_{\mathbb{C}^n} e^{-(H(z-y)(z-y))} f(y)\,dy,$$

for $f \in L^2(\mathbb{R}^n)$. Then, the map $B_A : L^2(\mathbb{R}^n) \to \mathcal{F}_A(\mathbb{C}^n)$ is a unitary isomorphism under the assumption that $\mathbb{R}^n$ is invariant under A.

Notice that, when A is the identity matrix I, then B_A turns out to be the classical Bargmann transform given by

$$B_I f(z) = Bf(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} e^{\frac{z^2}{2}} \int\limits_{\mathbb{R}^n} e^{-(z-x)^2} f(x)\,dx.$$

Our aim of this paper is to obtain a frame for $\mathcal{F}_A(\mathbb{C}^n)$ using a discretized version of reproducing kernel K_A. In fact, we show that there exist positive constants $a, b > 0$ such that $\{e^A_{ma+ilb}\colon\ m, l \in \mathbb{Z}^n\}$ is a frame in $\mathcal{F}_A$, where $e^A_w(z) = e^A(z,w) = e^{-\frac{1}{2}\,Re\langle Aw,w\rangle} K_A(z,w)$. We also show that there exist $a, b > 0$ such that the Gram operator associated with $\{e^A_{ma+ilb}\colon\ m, l \in \mathbb{Z}^n\}$ is bounded above and below. In the final part of the paper, we give a note on frames for Fock space associated with an operator $A \in B(\mathcal{H})$, where $B(\mathcal{H})$ denotes the set of all bounded linear operators on a Hilbert space $\mathcal{H}$.

2. The main results

First, we recall the definition of a frame.

Definition 2.1. Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{e_j\colon\ j \in J\}$ in H is called a frame if there exist positive constants C_1, C_2 such that for all $f \in \mathcal{H}$

$$C_1\|f\|^2 \leqslant \sum_{j\in J} \left|\langle f, e_j\rangle\right|^2 \leqslant C_2\|f\|^2. \tag{2.1}$$

Any two constants C_1, C_2 satisfying (2.1) are called frame bounds.

Let η be the unitary map on $L^2(\mathbb{R}^n)$ defined by

$$\eta(f) = (\det H)^{\frac{1}{2}} f\big(H^{\frac{1}{2}}.\big), \quad \forall f \in L^2\big(\mathbb{R}^n\big).$$

Let $\mathcal{B}_A$ denote the composition of B_A and η, *viz.*, $\mathcal{B}_A = B_A o \eta$. Then $\mathcal{B}_A$ is a unitary map of $L^2(\mathbb{R}^n)$ onto $\mathcal{F}_A(\mathbb{C}^n)$. Let $\mathcal{K}_A$ denote the reproducing kernel for the image of $L^2(\mathbb{R}^n)$ under $\mathcal{B}_A$. Then it can be easily shown that $\mathcal{K}_A = K_A$. Recall

$$e^A_w(z) = e^A(z,w) = e^{-\frac{1}{2}(Aw,w)} K_A(z,w), \quad z, w \in \mathbb{C}^n. \tag{2.2}$$

Our aim is to show that $\{e^A_{ma+ilb}\colon\ m, l \in \mathbb{Z}^n\}$ is frame in $\mathcal{F}_A(\mathbb{C}^n)$. In order to show this result, it is enough to show that

$$\big\{\mathcal{B}^*_A e^A_{ma+ilb}\colon\ m, l \in \mathbb{Z}^n\big\}$$

is frame in $L^2(\mathbb{R}^n)$.

The operator $\mathcal{B}_A$ can be explicitly written as

$$\mathcal{B}_A f(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \frac{(\det H)^{\frac{3}{4}}}{(\det_{\mathbb{R}} A)^{\frac{1}{4}}} e^{\frac{1}{2}(\langle Hz,\bar{z}\rangle + \langle z, Kz\rangle)} \int\limits_{\mathbb{R}^n} e^{-(\sqrt{H}z-u)^2} f(u)\,du.$$

A straightforward computation shows that the adjoint of $\mathcal{B}_A$ is given by

$$\mathcal{B}^*_A F(u) = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \frac{(\det H)^{\frac{3}{4}}}{(\det_{\mathbb{R}} A)^{\frac{1}{4}}} \pi^{-n} \Big(\det_{\mathbb{R}} A\Big)^{\frac{1}{2}} \int\limits_{\mathbb{C}^n} \overline{e^{\frac{1}{2}(\langle Hz,\bar{z}\rangle + \langle z, Kz\rangle)} e^{-(\sqrt{H}z-u)^2}} F(z) e^{-(Az,z)}\,dx\,dy. \tag{2.3}$$

In order to show that $\{\mathcal{B}_A^* e_{ma+ilb}^A\}$ is a frame in $L^2(\mathbb{R}^n)$, we need a few definitions. Before defining them, we give the following notations.

For $x=(x_1,x_2,\ldots,x_n)$, $y=(y_1,y_2,\ldots,y_n)$, $x_i, y_i \in \mathbb{R}^n$, $i=1,2,\ldots,n$, we denote $[x,y]=\prod_{i=1}^n [x_i,y_i]$ and $\bar{1}$ to be the n-vector given by $\bar{1}=(1,1,\ldots,1)$. The notation $x \leqslant y$ means that $x_i \leqslant y_i$ for all $i=1,2,\ldots,n$.

We shall define H-Wiener space associated with the operator H mentioned above.

Definition 2.2. The H-Wiener space is defined to be the collection of all $g \in L^\infty(\mathbb{R}^n)$ such that for some $a>0$,

$$\|g\|_{W,H,a} = \sum_{\mu\in\mathbb{Z}^n} \operatorname*{ess\,sup}_{x\in[0,H\cdot\bar{1}a]} \left| g\big(H^{-\frac{1}{2}}(x-H\mu a)\big)\right| < \infty.$$

Definition 2.3. The H-correlation function for $e^{-\|x\|^2}$ is defined by

$$G_m(x) = G_m^{H,a,b}(x) = \sum_{k\in\mathbb{Z}^n} e^{-\|H^{-\frac{1}{2}}(x-Hka)\|^2} e^{-\|H^{-\frac{1}{2}}(x-\frac{m}{b}-Hka)\|^2}.$$

Now, we shall show that $e^{-\|x\|^2}$ is in the H-Wiener space. Consider

$$\begin{aligned}
\left\| e^{-\|x\|^2} \right\|_{W,H,a} &= \sum_{\mu\in\mathbb{Z}^n} \operatorname*{ess\,sup}_{x\in[0,H\cdot\bar{1}a]} e^{-\|H^{-\frac{1}{2}}(x-H\mu a)\|^2} \\
&\leqslant \sum_{\mu\in\mathbb{Z}^n} \operatorname*{ess\,sup}_{x\in[0,H\cdot\bar{1}a]} e^{-\frac{1}{c}\|(x-H\mu a)\|^2} \\
&\leqslant \sum_{\mu\in\mathbb{Z}^n} \operatorname*{ess\,sup}_{x\in[0,H\cdot\bar{1}a]} e^{-\frac{1}{c}(\|x\|^2-2ad\|x\|\|\mu\|+a^2c^2\|\mu\|^2)} \\
&\leqslant \sum_{\mu\in\mathbb{Z}^n} e^{\frac{2ad}{c}c'\|\mu\|} e^{-a^2c\|\mu\|^2} \\
&= \sum_{\mu\in\mathbb{Z}^n} e^{\frac{2ad}{c}c'\|\mu\|} e^{-\frac{a^2c}{2}\|\mu\|^2} e^{-\frac{a^2c}{2}\|\mu\|^2} \\
&\leqslant C \sum_{\mu\in\mathbb{Z}^n} e^{-\frac{a^2c}{2}\|\mu\|^2} \\
&= C \sum_{\mu_1\in\mathbb{Z}} e^{-\frac{a^2c}{2}\mu_1{}^2} \sum_{\mu_2\in\mathbb{Z}} e^{-\frac{a^2c}{2}\mu_2{}^2} \cdots \sum_{\mu_n\in\mathbb{Z}} e^{-\frac{a^2c}{2}\mu_n{}^2} < \infty,
\end{aligned}$$

showing that $e^{-\|x\|^2}$ belongs to H-Wiener space. Here the constants c,d exist since H is bounded above and below, *viz.*, $c\|x\| \leqslant \|Hx\| \leqslant d\|x\|$ $\forall x\in\mathbb{R}^n$.

Proposition 2.4. *Let $a>0$. Then* $\lim_{b\to 0}\sum_{m\in\mathbb{Z}^n\setminus\{0\}} \|G_m\|_\infty = 0$.

Proof. We first observe that G_m is $H\cdot\bar{1}a\mathbb{Z}^n$ periodic. Then consider

$$\begin{aligned}
\sum_{m\neq 0\in\mathbb{Z}^n} \|G_m\|_\infty &= \sum_{m\neq 0\in\mathbb{Z}^n} \left\| \sum_{l\in\mathbb{Z}^n} e^{-\|H^{-\frac{1}{2}}(x-Hla)\|^2} e^{-\|H^{-\frac{1}{2}}(x-\frac{m}{b}-Hla)\|^2} \right\|_\infty \\
&= \sum_{m\neq 0\in\mathbb{Z}^n} \left\| \sum_{l\in\mathbb{Z}^n} e^{-\|H^{-\frac{1}{2}}(x-Hla)\|^2} e^{-\|H^{-\frac{1}{2}}(x-\frac{m}{b}-Hla)\|^2} \chi_{[0,H\cdot\bar{1}a]} \right\|_\infty \\
&= \sum_{m\neq 0\in\mathbb{Z}^n} \left\| \sum_{l\in\mathbb{Z}^n} e^{-\|H^{-\frac{1}{2}}(x-Hla)\|^2} \chi_{[0,H\cdot\bar{1}a]} e^{-\|H^{-\frac{1}{2}}(x-\frac{m}{b}-Hla)\|^2} \chi_{[0,H\cdot\bar{1}a]} \right\|_\infty.
\end{aligned}$$

Since $e^{-\|x\|^2}$ belongs to H-Wiener space, given $\epsilon>0$ there exists $N\in\mathbb{N}$ such that $\sum_{\|l\|>N} \operatorname{ess\,sup}_{x\in[0,H\cdot\bar{1}a]} e^{-\|H^{-\frac{1}{2}}(x-Hla)\|^2} < \epsilon$. Let $g_0(x) = e^{-\|H^{-\frac{1}{2}}(x)\|^2}\chi_{A_N}$, $g_1(x) = e^{-\|H^{-\frac{1}{2}}(x)\|^2}\chi_{A_N{}^c}$, where $A_N = \bigcup_{\|l\|\leqslant N}[Hla, H(l+\bar{1})a]$. Note that $\|g_1\|_{W,H,a} < \epsilon$. Let $T_m f(x) = f(x-m)$. Then

$$\begin{aligned}
\sum_{m\neq 0\in\mathbb{Z}^n}\|G_m\|_\infty &\leqslant \sum_{m\neq 0\in\mathbb{Z}^n}\left\|\sum_{l\in\mathbb{Z}^n} T_{Hla}e^{-\|H^{-\frac{1}{2}}(x)\|^2}\chi_{[0,H\cdot\bar{1}a]}T_{Hla+\frac{m}{b}}e^{-\|H^{-\frac{1}{2}}(x)\|^2}\chi_{[0,H\cdot\bar{1}a]}\right\|_\infty \\
&= \sum_{m\neq 0\in\mathbb{Z}^n}\left\|\sum_{l\in\mathbb{Z}^n} T_{Hla}(g_0+g_1)\chi_{[0,H\cdot\bar{1}a]}T_{Hla+\frac{m}{b}}(g_0+g_1)\chi_{[0,H\cdot\bar{1}a]}\right\|_\infty \\
&= \sum_{m\neq 0\in\mathbb{Z}^n}\left\|\sum_{l\in\mathbb{Z}^n} T_{Hla}(g_0)\chi_{[0,H\cdot\bar{1}a]}T_{Hla+\frac{m}{b}}(g_0)\chi_{[0,H\cdot\bar{1}a]}\right\|_\infty \\
&\quad + \sum_{m\in\mathbb{Z}^n}\left\|\sum_{l\in\mathbb{Z}^n} T_{Hla}(g_0)\chi_{[0,H\cdot\bar{1}a]}T_{Hla+\frac{m}{b}}(g_1)\chi_{[0,H\cdot\bar{1}a]}\right\|_\infty \\
&\quad + \sum_{m\neq 0\in\mathbb{Z}^n}\left\|\sum_{l\in\mathbb{Z}^n} T_{Hla}(g_1)\chi_{[0,H\cdot\bar{1}a]}T_{Hla+\frac{m}{b}}(g_0)\chi_{[0,H\cdot\bar{1}a]}\right\|_\infty \\
&\quad + \sum_{m\neq 0\in\mathbb{Z}^n}\left\|\sum_{l\in\mathbb{Z}^n} T_{Hla}(g_1)\chi_{[0,H\cdot\bar{1}a]}T_{Hla+\frac{m}{b}}(g_1)\chi_{[0,H\cdot\bar{1}a]}\right\|_\infty.
\end{aligned}$$

Choose b such that $\frac{1}{b} > \operatorname{diam}(A_N)$. Then in the right hand of the above inequality the first term becomes $\sum_{m\neq 0\in\mathbb{Z}^n}\|\sum_{l\in\mathbb{Z}^n} T_{Hla}(g_0)\chi_{[0,H\cdot\bar{1}a]}T_{Hla+\frac{m}{b}}(g_0)\chi_{[0,H\cdot\bar{1}a]}\|_\infty = 0$. Consider the second term. Choose a such that $H\cdot\bar{1}a \leqslant \frac{1}{b}\bar{1}$. Then the second term is

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^n}\left\|\sum_{l\in\mathbb{Z}^n} T_{Hla}(g_0)\chi_{[0,H\cdot\bar{1}a]}T_{Hla+\frac{m}{b}}(g_1)\chi_{[0,H\cdot\bar{1}a]}\right\|_\infty &\leqslant A_1\|g\|_{W,H,a}\|g_1\|_{W,H,a} \\
&\leqslant \epsilon A_1\|g\|_{W,H,a}.
\end{aligned}$$

Similarly the third term $\leqslant A_1\|g\|_{W,H,a}\|g_1\|_{W,H,a} \leqslant \epsilon A_1\|g\|_{W,H,a}$ and the fourth term $\leqslant \|g_1\|^2_{W,H,a} \leqslant \epsilon^2$. This proves our assertion. $\square$

Proposition 2.5. *There exist $C_1 > 0$, $0 < C_2 < \infty$ such that*

$$C_1 \leqslant \sum_l e^{-\|H^{-\frac{1}{2}}(x-Hla)\|^2} \leqslant C_2.$$

Proof. Since H is invertible, there exist $c > 0$, $0 < d < \infty$ such that $c\|x\| \leqslant \|Hx\| \leqslant d\|x\|$ $\forall x \in \mathbb{R}^n$. Consider

$$\begin{aligned}
\sum_l e^{-\|H^{-\frac{1}{2}}(x-Hla)\|^2} &= \sum_l e^{-\|H^{\frac{1}{2}}(H^{-1}x-la)\|^2} \\
&\leqslant \sum_l e^{-c\|(H^{-1}x-la)\|^2} \\
&= \sum_l e^{-c[(y_1-al_1)^2+\cdots+(y_n-al_n)^2]} \\
&= \sum_{l_1} e^{-c(y_1-al_1)^2}\sum_{l_2} e^{-c(y_2-al_2)^2}\cdots\sum_{l_n} e^{-c(y_n-al_n)^2} \\
&= \sum_{l_1} e^{-ca^2(\frac{y_1}{a}-l_1)^2}\sum_{l_2} e^{-ca^2(\frac{y_2}{a}-l_2)^2}\cdots\sum_{l_n} e^{-ca^2(\frac{y_n}{a}-l_n)^2} \\
&\leqslant \sum_{l_1} e^{-ca^2{l_1}^2}\sum_{l_2} e^{-ca^2{l_2}^2}\cdots\sum_{l_n} e^{-ca^2{l_n}^2} \\
&= \left(1+\sum_{l\in\mathbb{Z}} e^{-ca^2l^2}\right)^n \\
&\leqslant \left(1+2\sum_{l=1}^{\infty} e^{-ca^2l}\right)^n \\
&= \left(\frac{1+e^{ca^2}}{e^{ca^2}-1}\right)^n.
\end{aligned}$$

Again, consider

$$\begin{aligned}\sum_l e^{-\|H^{\frac{1}{2}}(H^{-1}x-la)\|^2} &\geqslant \sum_{l_1} e^{-da^2(\frac{y_1}{a}-l_1)^2}\sum_{l_2} e^{-da^2(\frac{y_2}{a}-l_2)^2}\cdots\sum_{l_n} e^{-da^2(\frac{y_n}{a}-l_n)^2}\\ &\geqslant e^{-\frac{da^2}{2}}e^{-\frac{da^2}{2}}\cdots e^{-\frac{da^2}{2}}\\ &= e^{-\frac{nda^2}{2}}.\end{aligned}$$

Thus

$$C_1 \leqslant \sum_l e^{-\|H^{-\frac{1}{2}}(x-Hla)\|^2} \leqslant C_2,$$

where $C_1 = e^{-\frac{nda^2}{2}}$, $C_2 = (\frac{1+e^{ca^2}}{e^{ca^2}-1})^n$ and $H^{-1}x = (y_1, y_2, \ldots, y_n)$. $\square$

Theorem 2.6. *Assume that $H\cdot\bar{1} \in \mathbb{R}^n_+ = \{x \in \mathbb{R}^n\colon\ x = (x_1, x_2, \ldots, x_n),\ x_i > 0,\ i = 1, 2, \ldots, n\}$. Then there exist constants $a, b > 0$ such that $\{e^A_{ma+ilb}\colon\ m, l \in \mathbb{Z}^n\}$ is a frame in $\mathcal{F}_A(\mathbb{C}^n)$.*

Proof. Fix $a, b > 0$. Define

$$\phi_{m,l} = \mathcal{B}^*_A e^A_{ma+ilb}.$$

Since $\mathcal{B}_A$ is a unitary operator, it is enough to show that $\{\phi_{m,l}\colon\ m, l \in \mathbb{Z}^n\}$ is a frame in $L^2(\mathbb{R}^n)$. Using Eq. (2.3), one can explicitly calculate $\phi_{m,l}$. It turns out to be

$$\phi_{m,l}(x) = e^{-iab\langle Km,l\rangle}e^{-iab\langle Hm,l\rangle}e^{-2i\langle x,\sqrt{H}lb\rangle}e^{-\|x-\sqrt{H}ma\|^2}.$$

Consider

$$\begin{aligned}\sum_m\sum_l\bigl|\langle f,\phi_{m,l}\rangle\bigr|^2 &= \sum_m\sum_l\biggl|\int_{\mathbb{R}^n} f(x)e^{iab\langle Km,l\rangle}e^{iab\langle Hm,l\rangle}e^{2i\langle x,\sqrt{H}lb\rangle}e^{-\|x-\sqrt{H}ma\|^2}\,dx\biggr|^2\\ &= \sum_m\sum_l\biggl|\int_{\mathbb{R}^n} f(x)e^{2i\langle\sqrt{H}x,lb\rangle}e^{-\|x-\sqrt{H}ma\|^2}\,dx\biggr|^2\\ &= \det H^{-\frac{1}{2}}\sum_m\sum_l\biggl|\int_{\mathbb{R}^n} f\bigl(H^{-\frac{1}{2}}x\bigr)e^{2i\langle x,lb\rangle}e^{-\|H^{-\frac{1}{2}}x-\sqrt{H}ma\|^2}\,dx\biggr|^2.\end{aligned}$$

Define $\psi_m(t) = \sum_k f(H^{-\frac{1}{2}}(x-\frac{k}{b}))e^{-\|H^{-\frac{1}{2}}(x-\frac{k}{b})-\sqrt{H}ma\|^2}$. Then ψ_m is $\frac{1}{b}\mathbb{Z}^n$ periodic. Thus,

$$\sum_l\biggl|\int_{[0,\frac{1}{b}\bar{1}]}\psi_m(t)e^{-2i\langle t,lb\rangle}\,dt\biggr|^2 = b^{-n}\int_{[0,\frac{1}{b}\bar{1}]}\bigl|\psi_m(t)\bigr|^2\,dt.$$

Hence

$$\begin{aligned}\sum_m\sum_l\bigl|\langle f,\phi_{m,l}\rangle\bigr|^2 &= \det H^{-\frac{1}{2}}b^{-n}\sum_m\int_{[0,\frac{1}{b}\bar{1}]}\biggl|\sum_k f\biggl(H^{-\frac{1}{2}}\biggl(x-\frac{k}{b}\biggr)\biggr)e^{-\|H^{-\frac{1}{2}}(x-\frac{k}{b}-Hma)\|^2}\biggr|^2\,dx\\ &= \det H^{-\frac{1}{2}}b^{-n}\sum_m\int_{[0,\frac{1}{b}\bar{1}]}\sum_l\overline{f\biggl(H^{-\frac{1}{2}}\biggl(x-\frac{k}{b}\biggr)\biggr)}e^{-\|H^{-\frac{1}{2}}(x-\frac{k}{b}-Hma)\|^2}\\ &\quad\times\sum_k f\biggl(H^{-\frac{1}{2}}\biggl(x-\frac{k}{b}\biggr)\biggr)e^{-\|H^{-\frac{1}{2}}(x-\frac{k}{b}-Hma)\|^2}\,dx\\ &= \det H^{-\frac{1}{2}}b^{-n}\sum_m\int_{\mathbb{R}^n}\overline{f\bigl(H^{-\frac{1}{2}}(x)\bigr)}e^{-\|H^{-\frac{1}{2}}(x-Hma)\|^2}\end{aligned}$$

$$\times \sum_k f\left(H^{-\frac{1}{2}}\left(x-\frac{k}{b}\right)\right)e^{-\|H^{-\frac{1}{2}}(x-\frac{k}{b}-Hma)\|^2}\,dx$$

$$= \det H^{-\frac{1}{2}} b^{-n} \sum_k \int_{\mathbb{R}^n} \overline{f\left(H^{-\frac{1}{2}}(x)\right)} f\left(H^{-\frac{1}{2}}\left(x-\frac{k}{b}\right)\right)$$

$$\times \sum_m e^{-\|H^{-\frac{1}{2}}(x-Hma)\|^2} e^{-\|H^{-\frac{1}{2}}(x-\frac{k}{b}-Hma)\|^2}\,dx$$

$$= \det H^{-\frac{1}{2}} b^{-n} \int_{\mathbb{R}^n} \left|f\left(H^{-\frac{1}{2}}(x)\right)\right|^2 \sum_m e^{-2\|H^{-\frac{1}{2}}(x-Hma)\|^2}\,dx$$

$$+ \det H^{-\frac{1}{2}} b^{-n} \sum_{k\neq 0} \int_{\mathbb{R}^n} \overline{f\left(H^{-\frac{1}{2}}(x)\right)} f\left(H^{-\frac{1}{2}}\left(x-\frac{k}{b}\right)\right)$$

$$\times \sum_m e^{-\|H^{-\frac{1}{2}}(x-Hma)\|^2} e^{-\|H^{-\frac{1}{2}}(x-\frac{k}{b}-Hma)\|^2}\,dx.$$

On the one hand, by Proposition 2.5 we have

$$C_1 \leqslant \sum_m e^{-\|H^{-\frac{1}{2}}(x-Hma)\|^2} \leqslant C_2.$$

On the other hand, it follows from Proposition 2.4 that

$$\lim_{b\to 0} \sum_{m\in\mathbb{Z}^n\setminus\{0\}} \|G_m\|_\infty = 0.$$

Thus, there exists $b_0 > 0$ such that

$$D_1\|f\|^2 \leqslant \sum_m \sum_l \left|\langle f, \phi_{m,l}\rangle\right|^2 \leqslant D_2\|f\|^2, \quad \forall f \in L^2\left(\mathbb{R}^n\right),$$

where

$$D_1 = \left(b_0^{-n} C_1 - b_0^{-n} \sum_{k\neq 0} \|G_k\|_\infty\right) > 0 \quad \text{and}$$

$$D_2 = \left(b_0^{-n} C_2 + b_0^{-n} \sum_{k\neq 0} \|G_k\|_\infty\right) < \infty.$$

See Theorem 4.1.5 of [17] for $H = I$ and $n = 1$. □

Remark 2.7. If $A = I$, then $H = I$. Thus, when $n = 1$, the condition $H \cdot \bar{1}a \leqslant \frac{1}{b}\bar{1}$ used in Proposition 2.4 leads to $ab \leqslant 1$, which is as expected from the classical work of I. Daubechies and A. Grossmann [4].

Next, we shall show that we can also find $a, b > 0$ such that the Gram operator associated with $\{e^A_{ma+ilb}\colon\ m, l \in \mathbb{Z}^n\}$ is bounded above and below.

Definition 2.8. Let $\mathcal{H}$ be a separable Hilbert space and $\{e_j\}_{j\in J}$ be a sequence of elements in $\mathcal{H}$. Then the matrix $\{\langle e_i, e_j\rangle\}_{i,j\in J}$ is called the Gram operator associated with $\{e_j\}_{j\in J}$.

Theorem 2.9. *There exist positive constants $a, b > 0$ such that the Gram operator associated with $\{e^A_{ma+ilb}\colon\ m, l \in \mathbb{Z}^n\}$ is bounded above and below.*

Proof. Fix $a, b > 0$. Let G denote the Gram operator associated with $\{e^A_{ma+ilb}\colon\ m, l \in \mathbb{Z}^n\}$. Recall $\phi_{m,l} = \mathcal{B}^*_A e^A_{ma+ilb}$ and $\mathcal{B}_A$ is a unitary map of $L^2(\mathbb{R}^n)$ onto $\mathcal{F}_A(\mathbb{C}^n)$. Thus, it is enough to show that the Gram operator associated with $\{\phi_{m,l}\colon\ m, l \in \mathbb{Z}^n\}$ is bounded above and below.

Let us introduce the analysis operator

$$S : L^2\left(\mathbb{R}^n\right) \to l^2\left(\mathbb{Z}^n \times \mathbb{Z}^n\right)$$

given by

$$\{S(f)\}_{m,l} = \{\langle f, \phi_{m,l}\rangle\}_{m,l}.$$

Now the Gram operator associated with $\{\phi_{m,l}\colon\ m, l \in \mathbb{Z}^n\}$ is given by SS^*. It is enough to show that SS^* is bounded above and below.

Notice that SS^* is a linear operator on $l^2(\mathbb{Z}^n \times \mathbb{Z}^n)$ such that

$$\{SS^*c\}_{m,l} = \sum_{m',l'} c_{m',l'} \langle \phi_{m',l'}, \phi_{m,l}\rangle,$$

where $(c) = (c_{m',l'})_{(m',l')\in\mathbb{Z}^n\times\mathbb{Z}^n}$.

We also have

$$\langle \phi_{m',l'}, \phi_{m,l}\rangle = \langle e^A_{m'a+l'b}, e^A_{ma+lb}\rangle_{\mathcal{F}_A}.$$

But for $w, w' \in \mathbb{C}^n$, we have

$$\langle e^A_w, e^A_{w'}\rangle_{\mathcal{F}_A} = e^{-\frac{1}{2}(Aw,w)} e^{-\frac{1}{2}(Aw',w')} K_A(w, w')$$

as $\int_{\mathbb{C}^n} K_A(z, w)\overline{K_A(z, w')}\, d\mu_A(z) = K_A(w, w')$. By using explicit expression for $K_A(w, w')$ using (1.1) and using the fact that $A = H + K$, one can write

$$\begin{aligned}\langle e^A_w, e^A_{w'}\rangle_{\mathcal{F}_A} &= C_A^{-2} e^{-\frac{1}{2}[(Hw,w)+(Kw,w)]} e^{-\frac{1}{2}[(Hw',w')+(Kw',w')]} \\ &\quad \times e^{\frac{1}{2}(Kw,w)} e^{-\frac{i}{2}\Im\overline{\langle Kw,w\rangle}} e^{\langle Hw,w'\rangle} e^{\frac{1}{2}(Kw',w')} e^{\frac{i}{2}\Im\langle Kw',w'\rangle}.\end{aligned}$$

After simplification, we get

$$\begin{aligned}\left|\langle e^A_w, e^A_{w'}\rangle_{\mathcal{F}_A}\right| &= C_A^{-2} e^{-\frac{1}{2}(Hw,w)} e^{-\frac{1}{2}(Hw',w')} e^{(Hw,w')} \\ &= C_A^{-2} e^{-\frac{1}{2}(\sqrt{H}w,\sqrt{H}w)} e^{-\frac{1}{2}(\sqrt{H}w',\sqrt{H}w')} e^{(\sqrt{H}w,\sqrt{H}w')} \\ &= C_A^{-2} e^{-\frac{1}{2}\|\sqrt{H}(w-w')\|^2},\end{aligned}$$

where C_A is given by $C_A = (\frac{\det_{\mathbb{R}}(A)}{\det_{\mathbb{R}}(H)})^{\frac{1}{4}}$. Thus,

$$\begin{aligned}\left|\langle \phi_{m,l}, \phi_{m',l'}\rangle\right| &= \left|\langle e^A_{ma+ilb}, e^A_{m'a+il'b}\rangle_{\mathcal{F}_A}\right| \\ &= C_A^{-2} e^{-\frac{a^2}{2}\|\sqrt{H}(m-m')\|^2} e^{-\frac{b^2}{2}\|\sqrt{H}(l-l')\|^2},\end{aligned} \tag{2.4}$$

from which it follows that

$$\sum_{m,l} \left|\langle \phi_{m,l}, \phi_{m',l'}\rangle\right| \leqslant C_A^{-2}\left(\coth\frac{k^2a^2}{2}\coth\frac{k^2b^2}{2}\right)^n, \quad \forall (m', l') \in \mathbb{Z}^{2n},$$

where k is such that $0 < k\|x\| \leqslant \|\sqrt{H}x\|$, for every $x \in \mathbb{R}^n$. It is well known that, if $(a_{j,k})_{j,k\in J}$ is an infinite matrix over the index set J such that

$$\sup_{j\in J}\sum_{k\in J} |a_{j,k}| \leqslant K_1,$$

$$\sup_{k\in J}\sum_{j\in J} |a_{j,k}| \leqslant K_2,$$

then the operator B defined by the matrix-vector multiplication $(Bc)_j = \sum_{k\in J} a_{j,k}c_k$ is bounded from $l^p(J)$ to $l^p(J)$ for $1 \leqslant p \leqslant \infty$. The operator norm of B is bounded by

$$\|B\| \leqslant K_1^{\frac{1}{p'}} K_2^{\frac{1}{p}}.$$

This is a part of Schur's test and we refer to [16] for details.

Thus it follows that SS^* is bounded above by

$$C_A^{-2}\left(\coth\left(\frac{k^2a^2}{2}\right)\coth\left(\frac{k^2b^2}{2}\right)\right)^n.$$

It remains to show that SS^* is bounded below. Consider

$$\begin{aligned}\langle SS^*c, c\rangle &= \langle S^*c, S^*c\rangle \\ &= \int_{\mathbb{R}^n} \Big|\sum_{m,l} c_{m,l}\phi_{m,l}(x)\Big|^2\, dx \\ &= \sum_{m,l}\sum_{m',l'} c_{m,l}\overline{c_{m',l'}}\langle \phi_{m,l}, \phi_{m',l'}\rangle \\ &= \sum_{m,l}\sum_{m',l'} c_{m,l}\overline{c_{m',l'}}\big\langle e^A_{ma+ilb}, e^A_{m'a+il'b}\big\rangle_{\mathcal{F}_A}.\end{aligned}$$

Since $\langle SS^*c, c\rangle$ is positive, $\langle SS^*c, c\rangle = |\langle SS^*c, c\rangle|$. Hence

$$\begin{aligned}\big|\langle SS^*c, c\rangle\big| &\geqslant C_A^{-2}\sum_{m,l}|c_{m,l}|^2 - \sum_{m,l}\sum_{(m',l')\neq(m,l)} |c_{m,l}||c_{m',l'}|\big|\big\langle e^A_{ma+ilb}, e^A_{m'a+il'b}\big\rangle_{\mathcal{F}_A}\big| \\ &= C_A^{-2}\sum_{m,l}|c_{m,l}|^2 - C_A^{-2}\sum_{m,l}\sum_{(m',l')\neq(m,l)} |c_{m,l}||c_{m',l'}|e^{-\frac{a^2}{2}\|\sqrt{H}(m-m')\|^2}e^{-\frac{b^2}{2}\|\sqrt{H}(l-l')\|^2}. \qquad (2.5)\end{aligned}$$

Now consider the second term in the right hand side of (2.5)

$$\sum_{m,l}\sum_{(m',l')\neq(m,l)} |c_{m,l}||c_{m',l'}|e^{-\frac{a^2}{2}\|\sqrt{H}(m-m')\|^2}e^{-\frac{b^2}{2}\|\sqrt{H}(l-l')\|^2}.$$

By applying Schwarz inequality to the above expansion twice, and using the fact that SS^* is a self-adjoint operator, we get

$$\begin{aligned}&\sum_{m,l}\sum_{(m',l')\neq(m,l)} |c_{m,l}||c_{m',l'}|e^{-\frac{a^2}{2}\|\sqrt{H}(m-m')\|^2}e^{-\frac{b^2}{2}\|\sqrt{H}(l-l')\|^2} \\ &\quad\leqslant \sum_{m,l}\sum_{(m',l')\neq(m,l)} |c_{m,l}|^2 e^{-\frac{a^2}{2}\|\sqrt{H}(m-m')\|^2}e^{-\frac{b^2}{2}\|\sqrt{H}(l-l')\|^2} \\ &\quad\leqslant \sum_{m,l}|c_{m,l}|^2\sum_{(m',l')\neq(m,l)} e^{-\frac{k^2a^2}{2}\|(m-m')\|^2}e^{-\frac{k^2b^2}{2}\|(l-l')\|^2}.\end{aligned}$$

Consider

$$\begin{aligned}&\sum_{(m',l')\neq(m,l)} e^{-\frac{k^2a^2}{2}\|(m-m')\|^2}e^{-\frac{k^2b^2}{2}\|(l-l')\|^2} \\ &\quad= \sum_{(m,l)\neq(0,0)} e^{-\frac{k^2a^2}{2}\|m\|^2}e^{-\frac{k^2b^2}{2}\|l\|^2} \\ &\quad= \sum_{m\in\mathbb{Z}^n\setminus\{0\}} e^{-\frac{k^2a^2}{2}\|m\|^2} + \sum_{l\in\mathbb{Z}^n\setminus\{0\}} e^{-\frac{k^2b^2}{2}\|l\|^2} + \sum_{m\in\mathbb{Z}^n\setminus\{0\}} e^{-\frac{k^2a^2}{2}\|m\|^2}\sum_{l\in\mathbb{Z}^n\setminus\{0\}} e^{-\frac{k^2b^2}{2}\|l\|^2}. \qquad (2.6)\end{aligned}$$

By a straightforward computation, one can show that

$$\begin{aligned}\sum_{p\in\mathbb{Z}^n\setminus\{0\}} e^{-\frac{k^2a^2}{2}\|p\|^2} &= \sum_{i=1}^n \binom{n}{i}\left(2\sum_{q=1}^\infty e^{-\frac{k^2a^2}{2}q^2}\right)^i \\ &\leqslant \sum_{i=1}^n \binom{n}{i}\left(2\sum_{q=1}^\infty e^{-\frac{k^2a^2}{2}q}\right)^i.\end{aligned}$$

Choose $a > 0$ such that $\frac{2}{e^{\frac{k^2a^2}{2}}-1} < \frac{1}{3}$. Then

$$\sum_{p\in\mathbb{Z}^n\setminus\{0\}} e^{-\frac{k^2a^2}{2}\|p\|^2} \leqslant 2\big(2^n-1\big)\frac{1}{e^{\frac{k^2a^2}{2}}-1}.$$

We can choose b in a similar way. Thus, we get,

$$(2.6) \leqslant \frac{2(2^n-1)}{e^{\frac{k^2a^2}{2}}-1} + \frac{2(2^n-1)}{e^{\frac{k^2b^2}{2}}-1} + \frac{2(2^n-1)}{e^{\frac{k^2a^2}{2}}-1}\frac{2(2^n-1)}{e^{\frac{k^2b^2}{2}}-1} < 1.$$

Then we have the following:

$$\langle SS^*c, c\rangle \geqslant \langle c, c\rangle C_A^{-2} A_{a,b}^{n,k},$$

where

$$A_{a,b}^{n,k} = 1 - \left(\frac{2(2^n-1)}{e^{\frac{k^2a^2}{2}}-1} + \frac{2(2^n-1)}{e^{\frac{k^2b^2}{2}}-1} + \frac{2(2^n-1)}{e^{\frac{k^2a^2}{2}}-1}\frac{2(2^n-1)}{e^{\frac{k^2b^2}{2}}-1}\right)$$

showing that SS^* is bounded below. □

3. A note on frames for Fock space associated with $A \in B(\mathcal{H})$

In this section, we make use of the following notations. If $\mathcal{H}$ is an infinite-dimensional separable Hilbert space with orthonormal basis $\{e_n\colon n \in \mathbb{N}\}$, then $\mathcal{H}_n$ will denote the linear span of $\{e_1, e_2, \ldots, e_n\}$. Let P_n denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$. Let $B(\mathcal{H})$ denote the class of bounded operators on $\mathcal{H}$. For $A \in B(\mathcal{H})$, we define $A_n = P_n A|_{\mathcal{H}_n}$. The operators A_n are known as finite sections or finite-dimensional truncations or Galerkin approximations of A. The matrix A_n with respect to the basis $\{e_1, e_2, \ldots, e_n\}$ consists of first n rows and n columns of A.

Let $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$. It consists of all sequences $(u_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} \|u_n\|_{\mathcal{H}_n}^2 < \infty.$$

Remark 3.1. Let $\{\psi_i^n\}_{i\in\mathbb{Z}}$ be an orthogonal set in $\mathcal{H}_n$. For each $i \in \mathbb{Z}$, define

$$e_{i,n} := \bigl(0, 0, \ldots, \overbrace{\psi_i^n}^{n\text{-th place}}, 0, 0, \ldots\bigr).$$

Then $\{e_{i,n}\colon\ i \in \mathbb{Z},\ n \in \mathbb{N}\}$ is an orthogonal set in $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$. In particular, if $\{\psi_i^n\}_{i\in\mathbb{Z}}$ is an orthonormal basis in $\mathcal{H}_n$ for each n, then one can show that $\{e_{i,n}\colon\ i \in \mathbb{Z},\ n \in \mathbb{N}\}$ is an orthonormal basis in $\mathcal{H}$ and

$$\|f\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{H}_n}^2 = \sum_n \sum_i \bigl|\langle f_n, \psi_i^n\rangle\bigr|^2,$$

for $f \in \mathcal{H}$.

Let V be an infinite-dimensional separable complex Hilbert space with an orthonormal basis $\{e_n\colon n \in \mathbb{N}\}$. Let $A : V \to V$ be a bounded symmetric, positive-definite real linear operator. Let $V_n \cong \mathbb{C}^n$ be the complex linear span of $\{e_1, e_2, \ldots, e_n\}$ and $V_{n,\mathbb{R}} \cong \mathbb{R}^n$, the real linear span of $\{e_1, e_2, \ldots, e_n\}$.

Let $A_n = P_n A|_{\mathbb{C}^n}$. Write $A_n = H_n + K_n$, where

$$H_n = \frac{A_n + J_n^* A_n J_n}{2}, \qquad K_n = \frac{A_n - J_n^* A_n J_n}{2}$$

with $J_n : \mathbb{C}^n \to \mathbb{C}^n$ defined by $J_n(z) = iz$.

Since A is symmetric and positive-definite as a real linear transformation, it follows that A_n is symmetric and positive-definite as a real transformation on $\mathbb{C}^n$. Further, H_n is self-adjoint, positive-definite and invertible relative to the Hermitian inner product.

The sequence of operators $\{\|\sqrt{H_n}\|\}$ is always uniformly bounded below by a positive number. In fact, since H, H_n are positive-definite and also invertible for each n, it is enough to show that $\|H_n^{-1}\|$ is uniformly bounded above by a positive number.

Recall the fact that if there exists an $n_0 \in \mathbb{N}$ such that S_n is invertible for all $n \geqslant n_0$, then the solutions of $S_n x^n = y_n$ lead to the solution x of $Sx = y$, where $x = \lim_n x^n$ iff S is invertible, for any $S \in \mathcal{B}(V)$. See, for example, [1].

Let $y \in V$. Then $y_n = y|_{V_n} = P_n y \in V_n$. As H and H_n are invertible, for all $n \in \mathbb{N}$, $\{H_n^{-1} y_n\}$ converges by the above fact. Thus there exists M_y such that $\|H_n^{-1} y_n\| \leqslant M_y$. Then it follows from uniform boundedness principle $\{\|H_n^{-1} P_n\|\}$ is uniformly bounded. On the other hand, $\|H_n^{-1} P_n\| = \|H_n^{-1}\|$. Thus $\{\|H_n^{-1}\|\}$ is uniformly bounded above by a positive number.

Let μ_{A_n} be the measure defined by $d\mu_{A_n}(z) = \pi^{-n}\sqrt{\det_{\mathbb{R}} A_n}\, e^{-Re\langle A_n z, z\rangle}\, dx\, dy$, $z \in \mathbb{C}^n$. Let $\mathcal{F}_{A_n}(\mathbb{C}^n)$ be the reproducing kernel Hilbert space consisting of holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ such that

$$\|F\|_{A_n}^2 := \int_{\mathbb{C}^n} \bigl|F(z)\bigr|^2\, d\mu_{A_n}(z) < \infty,$$

with the reproducing kernel, given by,

$$K_{A_n}(z,w) = C_{A_n}^{-2} e^{\frac{1}{2}\overline{\langle K_n z, z\rangle}} e^{\langle H_n z, w\rangle} e^{\frac{1}{2}\langle K_n w, w\rangle}$$

where $C_{A_n} = (\frac{\det_{\mathbb{R}}(A_n)}{\det_{\mathbb{R}}(H_n)})^{\frac{1}{4}}$. Assume that $A_n(\mathbb{R}^n) \subseteq \mathbb{R}^n$ for each $n \in \mathbb{N}$. Then $\mathcal{B}_{A_n}$ (defined as in Section 2, A replaced by A_n) is a unitary map between $L^2(\mathbb{R}^n)$ and $\mathcal{F}_{A_n}(\mathbb{C}^n)$.

Define

$$L^2_\infty = \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}^n)$$

and

$$\mathcal{F}_A = \bigoplus_{n=1}^{\infty} \mathcal{F}_{A_n}(\mathbb{C}^n).$$

This immediately leads to the following

Theorem 3.2. *The Bargmann transform $\mathcal{B}_A : L^2_\infty \to \mathcal{F}_A$, defined by*

$$(\mathcal{B}_A f)_n := (\mathcal{B}_{A_n} f_n)_n$$

is a unitary operator.

It follows from Theorem 2.6 that, for each $n \in \mathbb{N}$, there exist $a_n, b_n > 0$ such that $\{e^{A_n}_{la_n+imb_n}\colon l,m \in \mathbb{Z}^n\}$ is a frame in $\mathcal{F}_{A_n}$. In other words, there exist $a_n, b_n > 0$ such that

$$\begin{aligned} D_{n,1}\|F_n\|^2_{\mathcal{F}_{A_n}} &\leqslant \sum_{l,m\in\mathbb{Z}^n} \left|\left\langle F_n, e^{A_n}_{la_n+imb_n}\right\rangle_{\mathcal{F}_{A_n}}\right|^2 \\ &\leqslant D_{n,2}\|F_n\|^2_{\mathcal{F}_A}, \quad \forall F_n \in \mathcal{F}_{A_n}. \end{aligned} \tag{3.1}$$

Let $F = (F_1, F_2, \ldots, F_n, \ldots) \in \mathcal{F}_A$, with $F_n \in \mathcal{F}_{A_n}, n \in \mathbb{N}$. Taking sum over $n \in \mathbb{N}$ of (3.1) we get,

$$\sum_n D_{n,1}\|F_n\|^2_{\mathcal{F}_{A_n}} \leqslant \sum_n \sum_{l,m\in\mathbb{Z}^n} \left|\left\langle F_n, e^{A_n}_{la_n+imb_n}\right\rangle_{\mathcal{F}_{A_n}}\right|^2 \leqslant \sum_n D_{n,2}\|F_n\|^2_{\mathcal{F}_{A_n}}, \quad \forall F \in \mathcal{F}_A.$$

Suppose there exist constants C_1, C_2 such that $C_1 \leqslant D_{n,1}$, $C_2 \geqslant D_{n,2}$ $\forall n \in \mathbb{N}$. Then one can get

$$C_1 \sum_n \|F_n\|^2_{\mathcal{F}_{A_n}} \leqslant \sum_n \sum_{l,m\in\mathbb{Z}^n} \left|\left\langle F_n, e^{A_n}_{la_n+imb_n}\right\rangle_{\mathcal{F}_{A_n}}\right|^2 \leqslant C_2 \sum_n \|F_n\|^2_{\mathcal{F}_{A_n}}, \quad \forall F \in \mathcal{F}_A,$$

which will show that $\{\widetilde{e^{A_n}_{la_n+imb_n}}\colon l,m \in \mathbb{Z}^n,\ n \in \mathbb{N}\}$ will form a frame in $\mathcal{F}_A$, where

$$\widetilde{e^{A_n}_{la_n+imb_n}} := \big(0, 0, \ldots, 0, \overbrace{e^{A_n}_{la_n+imb_n}}^{n\text{-th place}}, 0, 0, \ldots\big) \in \mathcal{F}_A, \quad \forall l,m \in \mathbb{Z}^n,\ n \in \mathbb{N}.$$

However, the constants $D_{n,1}, D_{n,2}$ found in Theorem 2.6 of this paper are not sufficient enough to get such type of constants C_1 and C_2. It remains open to sharpen the bounds obtained in Theorem 2.6 so as to get the required constants C_1 and C_2.

Acknowledgments

We thank the referee for his valuable comments, remarks and useful suggestions. In fact, we owe the revised version completely to the referee as the revised version grew out of his deep question about the conditions on a, b from the earlier version of Theorem 2.6. We thank Professor S.H. Kulkarni for some useful discussions. We also thank National Board for Higher Mathematics, Department of Atomic Energy, India, Nos. 2/48(10)/2009-R & D II/1086 for the financial grant.

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